Null–Kähler structures, Symmetries and Integrability

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We feel honoured to be able to make this small contribution to the celebration of Jerzy Plebański's 75th birthday. Plebański has presumably regarded his work on complex relativity as a step towards producing general solutions to the Einstein equations on a real Lorentzian manifold. No one in the mid-seventies could expect that his contributions to the field would underlie the relation between twistor descriptions of anti-self-dual conformal structures and integrable models!

The main focus of this paper will be a $(2, 2)$ signature metric in Plebański's form

$$g = \mathrm{d}w\mathrm{d}x + \mathrm{d}z\mathrm{d}y - \Theta_{xx}\mathrm{d}z^2 - \Theta_{yy}\mathrm{d}w^2 + 2\Theta_{xy}\mathrm{d}w\mathrm{d}z. \tag{1}$$

Here (w, z, x, y) are local coordinates in an open ball in $\mathbb{R}^4$, and $\Theta : \mathbb{R}^4 \longrightarrow \mathbb{R}$ is an arbitrary real analytic function. Not all $(2, 2)$ inner–products can be put in this form even locally. To understand the local constraint imposed on g by (1) let us make the following

Definition 1 *A null-Kähler structure on a real four-manifold $\mathcal{M}$ consists of an inner product g of signature $(+ + − −)$ and a real rank-two endomorphism $N : T\mathcal{M} \to T\mathcal{M}$ parallel with respect to this inner product such that*

$$N^2 = 0, \qquad \textit{and} \qquad g(NX, Y) + g(X, NY) = 0$$

for all $X, Y \in T\mathcal{M}$.

Consider the splitting $T_{\mathbb{C}}\mathcal{M} \cong S_+ \otimes S_-$, where S_+ and S_- are complex two-dimensional spin bundles. The isomorphism $\Lambda^2{}_+(\mathcal{M}) \cong \mathrm{Sym}^2(S_+)$ between the bundle of self-dual two-forms and the symmetric tensor product of two spin bundles implies that the existence of a null–Kähler structure is in four dimensions equivalent to the existence of a parallel real spinor. The Bianchi identity implies the vanishing of the curvature scalar. Null–Kähler structures are special cases of conformally recurrent structures investigated in [20]. In [1] and [5] it was shown that null–Kähler structures are locally given by one arbitrary function of four variables, and admit a canonical form (1) with $N = \mathrm{d}w \otimes \partial/\partial y - \mathrm{d}z \otimes \partial/\partial x$.

Further conditions can be imposed on the curvature of g to obtain non–linear PDEs for the potential function Θ

$$\begin{array}{ccccc} & & \text{ASD} & & \\ & \nearrow & & \searrow & \\ \text{Null Kähler} & & & & \text{Heavenly.} \\ & \searrow & & \nearrow & \\ & & \text{Einstein} & & \end{array}$$

Define

$$f := \Theta_{wx} + \Theta_{zy} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 \quad (2)$$

- The Einstein condition implies that

$$f = xP(w, z) + yQ(w, z) + R(w, z), \quad (3)$$

  where P, Q and R are arbitrary functions of (w, z). In fact the number of the arbitrary functions can be reduced down to one by redefinition of Θ and the coordinates, but for reasons which will become clear in the last section we prefer the form (3). This is the hyper–heavenly equation of Plebański and Robinson [19] for non–expanding metrics of type $[N]\times$[Any]. (Recall that $(\mathcal{M}, g)$ is called hyper–heavenly if the self–dual Weyl spinor is algebraically special). The solvability of equation (3) will be discussed in the last section of this paper.

- The conformal anti–self–duality (ASD) condition implies a 4th order PDE for Θ

$$\Box f = 0, \quad (4)$$

  where $\Box$ is the Laplace–Beltrami operator defined by the metric g. This equation is integrable: It admits a Lax pair and its solutions can in principle be found by twistor methods [5].

- Imposing both conformal ASD and Einstein condition implies (possibly after a redefinition of Θ) that $f = 0$, which yields the celebrated second heavenly equation of Plebański [18]

$$\Theta_{wx} + \Theta_{zy} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 = 0. \quad (5)$$

Many lower–dimensional integrable systems can be obtained form (5) or (4) if the associated metrics admit symmetries. The analysis of symmetry reductions can be made coordinate independent and more systematic by introducing some geometry on the space of orbits of the Killing vector. The relevant structure consists of a conformal structure compatible with a torsion free connection. The constraints induced by (5) or (4) imply the Einstein–Weyl (EW) equations. The study of these equation goes back to Cartan [3]. In the next section they will be presented in a modern language of Hitchin [13]. We shall then review the special classes of EW spaces, and their relation to solutions of the heavenly equation (5). In the last section we shall address the question of integrability of the non–ASD equation (3). It will be shown that Null Kähler Einstein metrics with symmetry preserving the Null-Kähler form locally depend on solutions to the variable coefficient dispersionless Kadomtsev–Petviashvili equation.

1 Einstein–Weyl geometry and symmetry reductions

Let M be an n-dimensional manifold with a torsion-free connection D, and a conformal structure $[h]$ which is compatible with D in a sense that

$$Dh = \omega \otimes h$$

for some one-form ω. Here $h \in [h]$ is a representative metric in a conformal class. If we change this representative by $h \rightarrow \psi^2 h$, then $\omega \rightarrow \omega + 2\mathrm{d}\ln\psi$, where ψ is a non-vanishing function on M. The space of oriented D–geodesics in M is a manifold $\mathcal{Z}$ of dimension $2n-2$. It can be identified with a quotient space of the projectivised tangent bundle $P(TM)$ by the geodesic spray. There exists a fixed point free map $\tau : \mathcal{Z} \longrightarrow \mathcal{Z}$ which reverses an orientation of each geodesics.

To describe a tangent space to $\mathcal{Z}$ at the geodesic $\gamma(t)$ take a curve of geodesics $\gamma(s,t)$ with $\gamma(0,t) = \gamma(t)$ and consider the Jacobi vector field

$$V = \frac{\partial \gamma}{\partial s}|_{s=0}.$$

The $(2n-2)$ dimensional tangent space $T_\gamma \mathcal{Z}$ is then just the space of solutions to the Jacobi's equation

$$(D_U)^2 V + R(V,U)U = 0 \tag{6}$$

modulo vector fields tangent to γ. Here $U = \mathrm{d}\gamma/\mathrm{d}t$, and R is the curvature tensor of D defined by

$$R(U,V)W = [D_U, D_V]W - D_{[U,V]}W, \qquad U, V, W \in TM.$$

Note that in general the Ricci tensor constructed out of R is not symmetric, and its skew part is proportional to $\mathrm{d}\omega$.

Now consider the special case of three-dimensional Weyl manifolds, and define the almost-complex structure on $\mathcal{Z}$ by

$$J(V) = \frac{U \times V}{\sqrt{h(U,U)}},$$

where $\times$ is the usual vector product on $\mathbb{R}^3$.

If V is a Jacobi field orthogonal to U then $J^2 = -\mathrm{Id}$. The solution space to (6) is J–invariant if $([h], D)$ satisfy the conformally invariant Einstein–Weyl equations

$$R_{(ab)} = \frac{1}{3} r h_{ab}, \qquad a, b, ... = 1, 2, 3. \tag{7}$$

Here $R_{(ab)}$ is the symmetrised Ricci tensor of D, and r is the Ricci scalar. If equations (7) are satisfied, then J is automatically integrable. In fact we have the following result

Theorem 1 (Hitchin [13]) *There is a one-to-one correspondence between local solutions to the Einstein–Weyl equations* (7)*, and complex surfaces (twistor spaces) equipped with a fixed-point free anti-holomorphic involution* τ*, and a* τ*-invariant rational curve with a normal bundle* $\mathcal{O}(2)$*.*

The EW space can be completely reconstructed form the twistor data. Since $H^0(\mathbb{CP}^1, \mathcal{O}(2)) = \mathbb{C}^3$, and $H^1(\mathbb{CP}^1, \mathcal{O}(2)) = 0$ we can use Kodaira's theorem. The EW space is a space of those $\mathcal{O}(2)$ curves which are τ-invariant. The family of such curves passing through a given point (and its conjugate) is a geodesic of a Weyl connection of D. To construct a conformal structure $[h]$ consider a point on a τ-invariant $\mathcal{O}(2)$ curve L_p. This point represents a point in a sphere of directions $(T_pM - 0)/\mathbb{R}^+$, and the conformal structure on L_p induces a quadratic conformal structure in M.

1.1 Special shear–free geodesic congruences

Recall that a geodesic congruence Γ in a region in $\hat{M} \subset M$ is a set of geodesic, one through each point of $\hat{M}$. Let V be a generator of Γ (a vector field tangent to Γ). The geodesic condition $V^a D_a V^b \sim V^b$

implies $D_aV^b = M_a{}^b + A_aV^b$ for some A_a, where $M_a{}^b$ is orthogonal to V^a on both indices. Consider the decomposition of M_{ab}

$$M_{ab} = \Omega_{ab} + \Sigma_{ab} + \frac{1}{2}\theta\hat{h}_{ab}$$

The shear Σ_{ab} is trace-free and symmetric. The twist Ω_{ab} is anti-symmetric, and the divergence θ is is a weighted scalar.. Here $\hat{h}_{ab} = ||V||^2h_{ab} - V_aV_b$ is an orthogonal projection of h_{ab}. The shear-free geodesics congruences (SFC) exist on any Einstein–Weyl space. This follows from a three-dimensional version of Kerr's theorem which states that SFCs correspond to to holomorphic curves in $\mathcal{Z}$. On the other hand imposing conditions on twist and divergence of a congruence gives restrictions on EW structures, and can be used to reduce the EW equations to some known and new integrable equations. This method was first applied in [24]. The general theory of SFC and its relation to EW geometry was developed in [2].

- Vanishing of the twist of an SFC implies existence of a foliation of an EW space by surfaces orthogonal to the congruence. It follows from the shear-free condition that these surfaces are equipped with a conformal structure. The EW structure can be locally put in the form

$$h = e^U(\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}t^2, \qquad \omega = 2U_t\mathrm{d}t,$$

  Here (x, y) are isothermal coordinates on the surfaces, $\partial/\partial t$ is normal to the surfaces, and $U = U(x, y, t)$ is a function. The EW equations reduce [24] to the Boyer–Finley–Plebański (BFP) equation

$$U_{xx} + U_{yy} + (e^U)_{tt} = 0. \tag{8}$$

  The preferred congruence is given by $\mathrm{d}t$ in the above coordinates. The system of geodesics (x, y) =const equipped with two possible orientations becomes a pair of complex curves $\mathcal{D}$ and $\tau(\mathcal{D})$ in $\mathcal{Z}$. LeBrun [16] shows that the divisor class $\mathcal{D} + \tau(\mathcal{D})$ represents the line bundle $\kappa^{-1/2}$, where $\kappa \longrightarrow \mathcal{Z}$ is the canonical line bundle (the bundle of holomorphic two-forms).

- The existence of a parallel congruence implies [6] the existence of a local coordinate system such that

$$h = \mathrm{d}y^2 - 4\mathrm{d}x\mathrm{d}t - 4U\mathrm{d}t^2, \qquad \omega = -4U_x\mathrm{d}t,$$

  and the EW condition reduces to the dispersion-less Kadomtsev–Petviashvili (dKP) equation

$$(U_t - UU_x)_x = U_{yy}. \tag{9}$$

  If $U(x, y, t)$ is a smooth real function of real variables then the conformal structure has signature $(+ + -)$. The real structure τ on $\mathcal{Z}$ differs form the one considered in Theorem 1. Now τ fixes an equator on each $\mathbb{CP}^1$ and interchanges upper and lower hemisphere. One can verify that the vector $\partial/\partial x$ is a real null vector, covariantly constant in the Weyl connection, and with weight $-1/2$. Covariantly constant real null vector givers rise to a parallel real weighted spinor, and finally to a preferred section of $\kappa^{-1/4}$ in $\mathcal{Z}$.

- The existence of the divergence-free SFC implies [4] that locally the EW structure is given by

$$h = (\mathrm{d}y + U\mathrm{d}t)^2 - 4(\mathrm{d}x + W\mathrm{d}t)\mathrm{d}t, \qquad \omega = U_x\mathrm{d}y + (UU_x + 2U_y)\mathrm{d}t,$$

  where $U(x, y, t)$ and $W(x, y, t)$ satisfy a system of quasi-linear PDEs

$$U_t + W_y + UW_x - WU_x = 0, \qquad U_y + W_x = 0. \tag{10}$$

  The the preferred congruence $\mathrm{d}t$ is shear free, and its divergence $D * (\mathrm{d}t)$ vanishes. The corresponding twistor space $\mathcal{Z}$ fibres holomorphically over $\mathbb{CP}^1$ [2].

It is interesting to note that equations (8, 9, 10) are integrable in more than one sense as they possess infinitely many hydrodynamic reductions [9, 12].

1.2 Examples of solutions

Equations (8) and (9) are equivalent to

$$\mathrm{d} *_h \mathrm{d}U = 0, \tag{11}$$

where $*_h$ is the Hodge operator taken with respect to the corresponding EW conformal structure (equation (10) also implies (11), but the converse does not hold in general).

Equations which can be written in the form (11) may be reduced to ODEs by a 'central quadric' ansatz. The ansatz is to seek solutions constant on central quadrics or equivalently to seek a matrix $M_{ab}(U)$ so that a solution of (11) is determined implicitly by

$$M_{ab}(U)x^a x^b = C, \tag{12}$$

where $x^a = (x, y, t)$, and C =const. The general method of reducing this condition to an ODE is described in [8]. Although the ansatz leads to ODEs, the resulting solutions to (11) are not group invariant.

In the case of the BFP equation this ODE reduces to Painlevé III [24], and in the case of dKP the ODE reduces to Painlevé I or II [8]. The details of the dKP case are as follows:

- If $(M^{-1})_{33} \neq 0$ then (12) becomes

$$\begin{aligned} &x^2 v - y^2 w\,(wv - (\alpha - 1/2)) + \frac{1}{2}\,t^2 \left((\alpha - 1/2)^2 + 4\,wv\,(wv - (\alpha - 1/2)) + 2\,v^3\right) \\ &+xy\,(\alpha - 1/2) - ytv\,(\alpha - 1/2) - 2\,txv^2 = C(2wv - (\alpha - 1/2))^2, \end{aligned}$$

  where α is a constant parameter,

$$v = \frac{1}{2}\dot{w}(U) - w(U)^2 - U,$$

  and $w(U)$ satisfies Painlevé II

$$\frac{1}{4}\ddot{w} = 2w^3 + 2wU + \alpha.$$

- If $(M^{-1})_{33} = 0$ and $(M^{-1})_{23} \neq 0$ then (12) becomes

$$x^2 + w^2 y^2 - w\Big(\frac{\dot{w}^2}{4} - 4w^3\Big)t^2 - 4xtw^2 + 2wxy + \Big(\frac{\dot{w}^2}{4} - 4w^3\Big)yt = C\dot{w}^2,$$

  where $w(U)$ satisfies Painlevé I

$$\ddot{w}/4 = 6w^2 + 2U.$$

- Finally if $(M^{-1})_{33} = (M^{-1})_{23} = 0$ then dKP reduces to a linear equation.

1.3 Heavenly spaces with symmetry

A link between three-dimensional EW geometry and symmetries of the heavenly equation is provided by the following

Theorem 2 (Jones and Tod[14]) *Let $(\mathcal{M}, [g])$ be a real four-manifold with ASD conformal curvature, and a conformal non-null Killing vector. The space of trajectories of this vector is equipped with an EW structure defined by*

$$h := |K|^{-2} g \pm |K|^{-4} K \odot K, \qquad \omega := 2|K|^{-2} *_g (K \wedge \mathrm{d}K), \tag{13}$$

where $*_g$ is taken w.r.t some $g \in [g]$, K is the one-form dual to the conformal Killing vector, and $|K|^2 = g(K, K)$. All three-dimensional EW structures arise in this way. The $+$ and $-$ signs in (13) refer to the signature of $[g]$ being Euclidean or neutral respectively.

This result was improved in [2] and [6], where it was shown that all EW spaces can be obtained as reductions form scalar-flat Kähler, or hyper-complex four manifolds respectively.

If we assume that there exists $g \in [g]$ such that $(\mathcal{M}, g)$ is Ricci flat, so that g arises form a solution to the heavenly equation (5), then a connection with the special classes of EW spaces can be established:

- If the symmetry fixes all self–dual two forms then the heavenly equation reduces to the Laplace equation in three dimensions [11]. The metric is in the Gibbons–Hawking class, the resulting Einstein–Weyl structures are trivial, and their mini-twistor space is $T\mathbb{CP}^1$.

- If the symmetry rotates the self-dual two-forms, then its lift to the bundle of self-dual two-forms has a fixed point. If this point corresponds to a non-simple two–form then the heavenly equation reduces to the BFP equation (8) [11]. If the fixed two-form is simple, then the reduced equation is dKP (9) [6].

- If the symmetry is only conformal but it fixes the self-dual two-forms, the heavenly equation reduces to equation (10) [4]. More general conformal symmetries have been studied in [7].

2 Integrability of the Hyper–Heavenly equations?

Hyper–heavenly (HH) equations and their reduction do not enjoy the elegant twistor description [17] associated to the anti-self-duality, and they are believed not to be integrable. This may be true for general HH spaces, but the simplest HH space– the null Kähler Einstein equation (3)– shares an integrable root with the heavenly equation (5). To see it define $L = \Theta_{xx}, M = \Theta_{xy}, N = \Theta_{yy}$ and write a system of three equations resulting from differentiating (5) w.r.t xx, xy and yy. This system should be complemented by adding the integrability conditions $L_y = M_x, M_y = N_x$ which guarantee that L, M, N admit a potential Θ. The analogous procedure applied to (3) yields the same over-determined system. The difference arises when one chooses the constants of integration (function of two variables) leading to back to Θ.

There is more evidence of integrability associated to hyper–heavenly spaces: In [22, 23] it was demonstrated that all Riemannian HH spaces of type $[D]\times$[Any] can locally be found from solutions to the BFP equation (8). Note that in this case the existence of the Killing vector does not have to be imposed, but it follows form the field equations – a product of two spinors defining a type D solution is a Killing spinor, and a contracted covariant derivative of a Killing spinor is a Killing vector.

In this section we shall consider the natural one–symmetry reduction of the null–Kähler Einstein spaces (3) and show that the resulting PDE in three dimensions differs form the dKP (9) equation by a function of one variable.

Consider a symmetry K which preserves the metric g, as well as the nilpotent endomorphism N. The canonical form of such symmetry in the coordinates adopted to the metric (1) turns out to be $K = \partial/\partial w - 2w\partial/\partial y$. This is a special form of Killing vector for non-expanding HH spaces, and so it must be contained in the classification of [10] or [21]. The Killing equations yield $(\mathcal{L}_K\Theta)_{xx} = (\mathcal{L}_K\Theta)_{yy} = 0, (\mathcal{L}_K\Theta)_{xy} = 1$. They integrate to

$$\Theta = wxy + yA(w, z) + xB(w, z) + C(w, z) + G(x, z, y + w^2).$$

The function C is pure gauge and can be set to zero without loss of generality. Imposing (3) and reabsorbing one arbitrary function of z into R (which itself can be arbitrary) yields

$$R + w^2 - A_z - B_w = w^2\gamma(z), \qquad Q = 1 + \gamma(z), \qquad P = \delta(z)$$

(where $\gamma = \gamma(z)$ and $\delta = \delta(z)$ are some arbitrary functions), and a nonlinear equation

$$-u\gamma - x\delta + G_{zu} + G_{xx}G_{uu} - G_{xu}^2 = 0 \qquad \text{where} \qquad u = y + w^2.$$

Write this equation as a closed system

$$\begin{aligned} \mathrm{d}G &= G_u\mathrm{d}u + G_z\mathrm{d}z + G_x\mathrm{d}x, \\ 0 &= -(u\gamma(z) + x\delta(z))\mathrm{d}x \wedge \mathrm{d}z \wedge \mathrm{d}u - \mathrm{d}G_u \wedge \mathrm{d}x \wedge \mathrm{d}u - \mathrm{d}G_x \wedge \mathrm{d}G_u \wedge \mathrm{d}z. \end{aligned} \tag{14}$$

Now express the first equation as $\mathrm{d}(G - uG_u) = G_z\mathrm{d}z + G_x\mathrm{d}x - u\mathrm{d}G_u$, and perform a Legendre transform

$$p := G_u, \qquad u = u(z, x, p), \qquad H(z, x, p) := -G(z, x, u(z, x, p)) + pu(z, x, p).$$

The relation $\mathrm{d}H = H_z\mathrm{d}z + H_x\mathrm{d}x + H_p\mathrm{d}p$ implies $H_z = -G_z, H_x = -G_x, H_p = u$. Equation (14) yields

$$-(H_p\gamma(z) + x\delta(z))\mathrm{d}x \wedge \mathrm{d}z \wedge \mathrm{d}H_p - \mathrm{d}p \wedge \mathrm{d}x \wedge \mathrm{d}H_p + \mathrm{d}H_x \wedge \mathrm{d}p \wedge \mathrm{d}z = 0,$$

which is equivalent to $(\gamma(z)H_p + \delta(z)x)H_{pp} + H_{pz} + H_{xx} = 0$. Taking the p derivative of this equation and using $H_p = u$ gives

$$(-u_z - (\gamma(z)u + \delta(z)x)u_p)_p = u_{xx}.$$

If $\gamma = 0$ then the above equation is linear. If $\gamma(z) \neq 0$ for some z then we restrict the domain of z such that $\gamma \neq 0$ for all z, and define $U(z, x, p) = u(z, x, p) + x\delta(z)/\gamma(z)$. Finally rename the coordinates $T = -z, X = p, Y = x$. To sum up, the $U(1)$–invariant null Kähler Einstein condition (3) can be reduced to a single PDE

$$(U_T - \gamma(T)UU_X)_X = U_{YY}. \tag{15}$$

This can be regarded as a variable coefficient generalisation the dKP equation (9). (Compare this reduction with the ASD null Kähler condition (4) which reduces down to a pair of coupled integrable PDEs: the dKP and its linearisation [5].) Equation (15) admits many explicit solutions, and shares some 'integrable properties' of the dKP. The metric on the space of orbits of the symmetry can be easily expressed in terms of U and it derivatives, but the associated geometry is unclear. Its characterisation could shed more light on the question of integrability of the simple HH space (3).

It is just a beginning of the story, as symmetry reductions can be performed for all hyper-heavenly spaces. This motivates the following

Question *What geometric structure is induced on a space of orbits of a symmetry in a hyper-heavenly manifold?*

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